

Infinite Dimensional 3-Lie Algebras and Their Connections to Harish-Chandra Modules

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Abstract: In this paper we construct two kinds of infinite-dimensional 3-Lie algebras from a given commutative associative algebra, and show that they are all canonical Nambu 3-Lie algebras. We relate their inner derivation algebras to Witt algebras, and then study the regular representations of these 3-Lie algebras and the natural representations of the inner derivation algebras. In particular, for the second kind of 3-Lie algebras, we find that their regular representations are Harish-Chandra modules, and the inner derivation algebras give rise to intermediate series modules of the Witt algebras and contain the smallest full toroidal Lie algebras without center.

Key words: 3-Lie algebra, Intermediate series module, Witt algebra, Toroidal Lie algebra, Inner derivation algebra.

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1 Introduction

Nambu [11] initiated implicitly the study of 3-Lie algebraic structures for generalized Hamiltonian mechanics by proposing the *canonical Nambu bracket* for a triple of classical observables on a three dimensional phase space with coordinates x, y, z

$$[f_1, f_2, f_3]_{\partial} = \frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)} \quad (1)$$

where the right-hand side is the Jacobian determinant of the vector function (f_1, f_2, f_3) .

As a generalization of Hamiltonian mechanics, Nambu mechanics attracts attention from both mathematicians and physicists. In physics, Takhtajan [16] described some basic principles of the formalism of Nambu mechanics, and introduced the notion of *Nambu-Poisson manifold*. Bayen and Flato studied Hamiltonian embedding for Nambu mechanics using Dirac's singular formalism [5]. Mukunda and Sudarshan investigated the relationship between Nambu and Hamiltonian mechanics [12]. Hirayama studied the realization problem of Nambu mechanics [8].

In mathematics, Filippov [7] originated systematically the theory of *abstract n -Lie algebras*, generalizing both Lie algebras and the canonical Nambu bracket to a more abstract stage. In particular, he studied the n -Lie algebra consisting of polynomials of n -variables with respect to the canonical Nambu bracket (1).

We call an abstract 3-Lie algebra a *canonical Nambu 3-Lie algebra* if it is isomorphic to a 3-Lie algebra whose bracket is defined by the canonical Nambu bracket (1).

A natural and important question is how to determine whether a given abstract 3-Lie algebra is a canonical Nambu 3-Lie algebra? A similar question arises for abstract n -Lie algebras. For $n \geq 3$, Alexeevsky and Guha [1] and Marmo-Vilasi-Vinogradov [10] showed that the n -Lie algebra structure on a given Nambu-Poisson manifold can be realized locally by the canonical Nambu bracket. In other words, there exists a local chart (coordinates) x_1, \dots, x_n in a neighbourhood of a point, with certain conditions, in the manifold such that the corresponding Poisson bracket of this n -Lie algebra is given by (1). However, there is still a long way to go to answer the question globally and completely.

Though the question is difficult to answer, there is evidence in the literature showing that the question is related to the realization problem of 3-Lie algebras. For instance, using the traditional generating method, Ding *et al.* [6] constructed a family of 3-Lie algebras, including the w_∞ 3-Lie algebra and $SDiff(T^2)$. Using the present terminology, these 3-Lie algebras are canonical Nambu 3-Lie algebras. Bai *et al.* [2] realized some 3-Lie algebras using two-dimensional extensions of metric Lie algebras; it is not clear if the bracket of these 3-Lie algebras can be realized by (1). Pozhidaev [7, 13, 14, 15] obtained some simple 3-Lie algebras; it is unknown whether these 3-Lie algebras are canonical Nambu 3-Lie algebras.

In this paper, we will shed some light on this question for 3-Lie algebras. Gathering some preliminary definitions and notation about n -Lie algebras in section 2, we then in section 3 construct an infinite-dimensional 3-Lie algebra from a given commutative associative algebra A using a derivation of A . We find that this 3-Lie algebra is a solvable canonical Nambu 3-Lie algebra. Its regular representation is a weight module, but not a Harish-Chandra module. Furthermore, the derivation algebra of this 3-Lie algebra is a direct sum of a Witt algebra W and an irreducible W -module.

In section 4, we define another infinite-dimensional 3-Lie algebra A_ω^δ from the same commutative associative algebra A using both an involution and a different derivation of A . We show that this 3-Lie algebra is a simple canonical Nambu 3-Lie algebra. Moreover, it contains a Lie subalgebra isomorphic to a Witt algebra.

In section 5, we prove that the inner derivation algebra of A_ω^δ is a direct sum of a Witt algebra W and the smallest full toroidal Lie algebra V without center. Moreover, the Lie algebra V is a direct sum of three isomorphic irreducible W -modules.

In section 6, we show that the regular representation of A_ω^δ is a Harish-Chandra module, and that the natural representation of the inner derivation algebra of A_ω^δ is an intermediate series module.

From now on, let \mathbb{F} be a field of characteristic zero unless stated otherwise, and let \mathbb{Z} be the set of integers.

2 Preliminaries

We collect some basic notation and facts about n -Lie algebras. A vector space L over a field \mathbb{F} is an arbitrary n -Lie algebra [1] if there is an n -ary multilinear operation $[\cdot, \dots, \cdot]$ satisfying the following anticommutative identity (2) and the generalized Jacobi identity (3)

$$[u_1, \dots, u_n] = (-1)^{\epsilon(\sigma)} [u_{\sigma(1)}, \dots, u_{\sigma(n)}], \quad (2)$$

$$[[u_1, \dots, u_n], v_2, \dots, v_n] = \sum_{i=1}^n [u_1, \dots, [u_i, v_2, \dots, v_n], \dots, u_n], \quad (3)$$

where $u_i, v_i \in L$ and σ runs over the symmetric group, and the number $\epsilon(\sigma)$ is the parity of the permutation σ . It is clear that 2-Lie algebras are the regular Lie algebras.

Let L be an arbitrary n -Lie algebra. A derivation of L is a linear map $\delta : L \rightarrow L$, such that

$$\delta([u_1, \dots, u_n]) = \sum_{i=1}^n [u_1, \dots, \delta(u_i), \dots, u_n], \quad \text{for all } u_1, \dots, u_n \in L.$$

The set $\text{Der}L$ of all derivations of L is a Lie subalgebra of the general Lie algebra $\text{gl}(L)$. This subalgebra is called the derivation algebra of L .

The map $\text{ad}(u_1, \dots, u_{n-1}) : L \rightarrow L$ defined by

$$\text{ad}(u_1, \dots, u_{n-1})(u_n) = [u_1, \dots, u_n], \quad \text{for all } u_1, \dots, u_n \in L.$$

is called a left multiplication of L . Clearly, each left multiplication is a derivation. The set $\text{ad}L$ of all finite linear combinations of left multiplications is an ideal of $\text{Der}L$. Each element of $\text{ad}(L)$ is referred to as an inner derivation of L . If $u_1, \dots, u_{n-1}, v_1, \dots, v_{n-1} \in L$, then

$$[\text{ad}(u_1, \dots, u_{n-1}), \text{ad}(v_1, \dots, v_{n-1})] = \sum_{i=1}^{n-1} \text{ad}(v_1, \dots, [u_1, \dots, u_{n-1}, v_i], \dots, v_{n-1}).$$

If L_1, L_2, \dots, L_n are subalgebras of L , let $[L_1, L_2, \dots, L_n]$ denote the subspace of L generated by all vectors $[u_1, u_2, \dots, u_n]$, where $u_i \in L_i$ for $i = 1, \dots, n$. Let $L^1 = [L, L, \dots, L]$ with n copies of L . Then L^1 is a subalgebra of L , called the derived algebra of L . We say that L is abelian if its derived algebra is zero.

An ideal of L is a subspace I such that $[I, L, \dots, L] \subseteq I$. We say that L is simple if L is not abelian and it has no ideals other than 0 and itself. An ideal I of L is a solvable ideal, if $I^{(r)} = 0$ for some $r \geq 0$, where $I^{(0)} = I$ and $I^{(s)}$ is defined by induction,

$$I^{(s+1)} = [I^{(s)}, I^{(s)}, L, \dots, L]$$

for $0 \leq s \leq r$. If L is a solvable ideal of itself, then L is called a solvable n -Lie algebra. An ideal I of L is called nilpotent, if $I^r = 0$ for some $r \geq 0$, where $I^0 = I$ and I^s is defined by

$$I^{s+1} = [I^s, I, L, \dots, L]$$

for $0 \leq s \leq r$. If L is a nilpotent ideal of itself, we call L a nilpotent n -Lie algebra. An ideal I of L is referred to as hypo-nilpotent, if I is a nilpotent n -Lie algebra, but it is not a nilpotent ideal of L .

A Cartan subalgebra of L is a nilpotent subalgebra of L such that for all $y \in L$, if $[y, H, L, \dots, L] \subseteq H$, then $y \in H$.

Lemma 2.1 ([2]) *Let $(L, [,])$ be a Lie algebra over a field \mathbb{F} , and let L^* be the dual space of L . Suppose $f \in L^*$ satisfies $f([u, v]) = 0$, for all $u, v \in L$. Then $(L, [, ,]_f)$ is a 3-Lie algebra with respect to the multiplication*

$$[u, v, w]_f = f(u)[v, w] + f(v)[w, u] + f(w)[u, v], \quad \text{for all } u, v, w \in L.$$

Lemma 2.2 ([4]) *Let δ be a derivation and ω be an involution of a commutative associative algebra A over \mathbb{F} . If $\omega\delta + \delta\omega = 0$, then A is a 3-Lie algebra under the multiplication*

$$[u, v, w] = \det \begin{pmatrix} \omega(u) & \omega(v) & \omega(w) \\ u & v & w \\ \delta(u) & \delta(v) & \delta(w) \end{pmatrix}, \quad \text{for all } u, v, w \in A. \quad (4)$$

To describe the connection of the structures of the 3-Lie algebras in this paper to the representations of Witt algebras, we need some basic facts about the Witt algebras. A basis for a Witt algebra W is given by the vector fields $L_n = -z^{n+1} \frac{\partial}{\partial z}$, for n in \mathbb{Z} . The Lie bracket of two vector fields is given by

$$[L_i, L_j] = (i - j)L_{i+j}.$$

This algebra has a central extension called the Virasoro algebra that is important in conformal field theory and string theory. A module V over the Witt algebra W is referred to as a weight module if it admits a weight space decomposition $V = \bigoplus_{\lambda \in \mathbb{F}} V_\lambda$ where

$$V_\lambda = \{v \in V \mid L_0 v = \lambda v\}.$$

A weight module over W is an intermediate series module if the dimension of every weight space is less than or equal to one.

Let L be a 3-Lie algebra and V be a vector space over \mathbb{F} . An L -module is a pair (V, ρ) where ρ is a linear mapping $\rho : L^{\wedge 2} \rightarrow \text{End}(V)$ satisfying

$$\begin{aligned} \rho([x_1, x_2, x_3], y) &= \rho(x_3, x_1)\rho(x_2, y) + \rho(x_2, x_3)\rho(x_1, y) + \rho(x_1, x_2)\rho(x_3, y) \\ [\rho(x_1, x_2), \rho(y_1, y)] &= \rho(x_1, x_2)\rho(y_1, y) - \rho(y_1, y)\rho(x_1, x_2) \\ &= \rho(y_1, [x_1, x_2, y]) + \rho([x_1, x_2, y_1], y) \end{aligned}$$

for all $x_i, y_1, y \in L$, $i = 1, 2, 3$.

A module (V, ρ) of a 3-Lie algebra L is referred to as a *weight module* if it admits a weight space decomposition $V = \bigoplus_{\lambda \in (H \wedge H)^*} V_\lambda$, where H is a Cartan subalgebra of L , $\lambda : H \wedge H \rightarrow \mathbb{F}$ is a linear mapping, and

$$V_\lambda = \{v \in V \mid \rho(h_1, h_2)v = \lambda(h_1, h_2)v, \quad \text{for all } h_1, h_2 \in H\}.$$

A module of L is a *Harish-Chandra* module if it is a weight module and every weight space is finite-dimensional. A Harish-Chandra module of L is an *intermediate series module* if the dimension of every weight space is less than or equal to one.

3 The 3-Lie algebra $(A, [,],_{f,k})$

Let A be a commutative associative algebra with a basis $\{L_r, M_r \mid r \in \mathbb{Z}\}$ and the multiplication

$$L_r L_s = L_{r+s}, \quad M_r M_s = M_{r+s}, \quad L_r M_s = 0, \quad \text{for all } r, s \in \mathbb{Z}. \quad (5)$$

For each $k \in \mathbb{Z}$, we define a derivation d_k of A by

$$d_k(aL_r + bM_t) = arL_{k+r},$$

for all $a, b \in \mathbb{F}$, $r, t \in \mathbb{Z}$. Through d_k , each integer k induces a Lie algebra structure $[,]_k$ on A defined by

$$[u, v]_k = d_k(u)v - ud_k(v), \quad \text{for all } u, v \in A. \quad (6)$$

We will show that each integer k gives rise to a canonical Nambu 3-Lie algebra structure on A .

To simplify notation and expressions, from now on, we assume that any bracket not listed in the multiplication of a 3-Lie algebra is always zero.

Theorem 3.1 *Let $f : A \rightarrow \mathbb{F}$ be a nonzero linear function with $f(L_r) = 0$ for all $r \in \mathbb{Z}$. Then for any integer k , A is a canonical Nambu 3-Lie algebra in the multiplication $[,],_{f,k}$ defined by*

$$[L_r, L_s, M_t]_{f,k} = f(M_t)(r - s)L_{r+s+k}, \quad \text{for all } L_r, L_s, M_t \in A. \quad (7)$$

Furthermore, A is a non-solvable 3-Lie algebra.

Proof. From the definition of f and the Lie structure (6) of A it follows that $f([A, A]) = 0$. By Lemma 2.1 we know that A is a 3-Lie algebra under the multiplication (7).

Now we show that this 3-Lie algebra is a canonical Nambu 3-Lie algebra. It suffices to show that it can be realized as a canonical Nambu 3-Lie algebra. Let $\beta_r = f(M_r)$, and choose the basis elements L_r and M_r of A to be

$$L_r = z \exp(rx), \quad M_r = \beta_r y \exp(kx), \quad \text{for all } r \in \mathbb{Z},$$

where \exp is the exponential function. Applying the Nambu bracket $[,],_\partial$ defined in (1) to these basis elements, we obtain

$$[L_r, M_s, M_t]_\partial = [M_r, M_s, M_t]_\partial = [L_r, L_s, L_t]_\partial = 0,$$

and

$$[L_r, L_s, M_t]_{\partial} = \begin{vmatrix} rz \exp(rx) & 0 & \exp(rx) \\ sz \exp(sx) & 0 & \exp(sx) \\ \beta_t y \exp(kx) & \beta_t \exp(kx) & 0 \end{vmatrix} = \beta_t(r-s)L_{r+s+k}.$$

So, the Nambu bracket coincides with the bracket $[\cdot, \cdot]_{f,k}$ on A . In other words, the 3-Lie algebra A with the bracket $[\cdot, \cdot]_{f,k}$ can be realized as the following canonical Nambu 3-Lie algebra

$$A = \sum_{r \in \mathbb{Z}} (\mathbb{F}z \exp(rx)) \oplus \sum_{r \in \mathbb{Z}} (\mathbb{F}y \exp(rx)).$$

Next, a direct computation yields that $A^1 = \sum_{r \in \mathbb{Z}} \mathbb{F}L_r$, and $[A^1, A^1, A^1]_{f,k} = 0$, but $A^{(s+1)} = [A^{(s)}, A^{(s)}, A]_{f,k} = A^1 \neq 0$ for all $s \geq 1$. It follows that A is a non-solvable 3-Lie algebra. \square

Corollary 3.2 *The subalgebra $L = \sum_{s \in \mathbb{Z}} \mathbb{F}L_s$ is a hypo-nilpotent minimal ideal of A , and $M = \sum_{s \in \mathbb{Z}} \mathbb{F}M_s$ is an abelian subalgebra.*

Corollary 3.3 *The regular representation of the 3-Lie algebra A is a weight module, but not a Harish-Chandra module.*

Proof. Let $L = A$ be the 3-Lie algebra $(A, [\cdot, \cdot]_{f,k})$. Then the Cartan subalgebra of L is

$$H = \mathbb{F}L_{-k} \oplus \sum_{t \in \mathbb{Z}} \mathbb{F}M_t,$$

which is an abelian subalgebra. The regular representation of L is (V, ad) , where $V = A$ as a vector space, and $\text{ad} : L \wedge L \rightarrow \text{End}(V)$ is given by

$$\text{ad}(x, y)(z) = [x, y, z]_{f,k} \quad \text{for all } x, y \in L \text{ and } z \in V.$$

Define $\lambda_r : H \wedge H \rightarrow \mathbb{F}$ by

$$\lambda_r(L_{-k}, M_t) = f(M_t)(r+k), \quad \lambda_r(M_s, M_t) = 0, \quad \text{for all } r \in \mathbb{Z}.$$

Then $V = \sum_{\lambda_r} V_{\lambda_r}$ where $V_{\lambda_r} = \mathbb{F}L_r$ if $r \neq 0$, and $V_0 = H$. But, $\dim H$ is not finite. \square

We now describe the inner derivation algebra $\text{ad}A$ of $(A, [\cdot, \cdot]_{f,k})$. We begin by introducing three types of left multiplications of $(A, [\cdot, \cdot]_{f,k})$,

$$W_{r,s} = \text{ad}(L_r, M_s), \quad X_{r,s} = \text{ad}(L_r, L_s), \quad Y_{r,s} = \text{ad}(M_r, M_s), \quad \text{for all } r, s \in \mathbb{Z}.$$

Define two subspaces W and X of $\text{ad}A$ by

$$W = \sum_{r \in \mathbb{Z}} \mathbb{F}W_{r,0} \quad \text{and} \quad X = \mathbb{F}X_{1,-1} + \sum_{r \in \mathbb{Z}, r \neq 0} \mathbb{F}X_{r,0}.$$

Theorem 3.4 *The Lie subalgebra W of $\text{ad}A$ is isomorphic to a Witt algebra, and X is an irreducible W -module. Furthermore,*

$$\text{ad}A = W \oplus X$$

is not solvable, and X is the only minimal ideal of $\text{ad}A$.

Proof. Let $\beta_s = f(M_s)$. Using (7), it is routine to check that $Y_{r,s} = 0$ and, for all $r, s, t \in \mathbb{Z}$,

$$\begin{aligned} W_{r,s}(L_t) &= \beta_s(t-r)L_{r+t+k}, \\ W_{r,s}(M_t) &= X_{r,s}(L_t) = 0, \\ X_{r,s}(M_t) &= \beta_t(r-s)L_{r+s+k}, \\ Y_{r,s}(M_t) &= Y_{r,s}(L_t) = 0. \end{aligned}$$

Since $f \neq 0$, there exists $s_0 \in \mathbb{Z}$ such that $\beta_{s_0} \neq 0$. A direct computation shows that if $s \neq -r$, then

$$X_{r,s} = \frac{r-s}{r+s} X_{r+s,0}.$$

If $s = -r$, then $X_{r,s} = \frac{1}{r} X_{1,-1}$. Hence $W_{r,s} = \frac{\beta_s}{\beta_0} W_{r,s_0}$ and $X_{r,s} = -X_{s,r}$, for $r, s \in \mathbb{Z}$. It follows that

$$\{W_{s,s_0}, X_{r,0}, X_{1,-1} \mid r, s \in \mathbb{Z}, r \neq 0\}$$

is a basis of the inner derivation algebra $\text{ad}(A)$. With (7), we obtain

$$[W_{r,s_0}, W_{s,s_0}] = \beta_{s_0}(s-r)W_{s+r+k,s_0}.$$

Thus $W = \sum_{s \in \mathbb{Z}} \mathbb{F}W_{s,s_0}$ is isomorphic to a Witt algebra. Thus $\text{ad}A$ is not solvable.

Without showing the details, a moderate computation gives rise to $[X_{r,0}, X_{s,0}] = [X_{r,0}, X_{1,-1}] = 0$, and

$$[W_{s,s_0}, X_{1,-1}] = \begin{cases} 2\beta_{s_0} \frac{t-s}{s+k} X_{s+k,0}, & s \neq -k, \\ 2\beta_{s_0} \frac{1}{s} X_{-1,1}, & s = -k, \end{cases}$$

as well as

$$[W_{s,s_0}, X_{r,0}] = \begin{cases} r\beta_{s_0} \frac{r-s+k}{r+s+k} X_{r+s+k,0}, & r+s \neq -k, \\ r\beta_{s_0} \frac{1}{s} X_{-1,1}, & r+s = -k, s \neq 0, \\ r\beta_{s_0} X_{r+k,0}, & r = -k, s = 0. \end{cases}$$

It follows that $X = FX_{1,-1} + \sum_{r \in \mathbb{Z}, r \neq 0} FX_{r,0}$ is an irreducible W -module. Moreover, $\text{ad}A = W \oplus X$, and X is its only minimal ideal. \square

4 The 3-Lie algebra A_ω^δ

Again, let A be the commutative associative algebra defined by (5). Define the derivation δ of A by

$$\delta(L_r) = rL_r, \quad \delta(M_r) = rM_r, \quad \text{for all } r \in \mathbb{Z}. \quad (8)$$

Let $\omega : A \rightarrow A$ be a linear mapping given by

$$\omega(L_r) = M_{-r}, \quad \omega(M_r) = L_{-r}, \quad \text{for all } r \in \mathbb{Z}. \quad (9)$$

Then ω is an involution of A , since $\omega^2 = \text{id}$ and

$$\begin{aligned} \omega(L_r L_s) &= \omega(L_{r+s}) = M_{-r-s} = M_{-r} M_{-s} = \omega(L_r) \omega(L_s), \text{ and} \\ \omega(M_r M_s) &= \omega(M_{r+s}) = L_{-r-s} = L_{-r} L_{-s} = \omega(M_r) \omega(M_s). \end{aligned}$$

Furthermore,

$$(\delta\omega + \omega\delta)(L_r) = (\delta\omega + \omega\delta)(M_r) = 0, \quad \text{for all } r \in \mathbb{Z}. \quad (10)$$

We have the following result.

Theorem 4.1 *A is a simple canonical Nambu 3-Lie algebra under the multiplication $[\cdot, \cdot, \cdot]_\omega$ defined by, for all $r, s, t \in \mathbb{Z}$,*

$$[L_r, L_s, M_t]_\omega = (s - r)L_{r+s-t}, \quad (11)$$

$$[L_r, M_s, M_t]_\omega = (t - s)M_{s+t-r}. \quad (12)$$

Proof. Since the derivation δ and involution ω satisfy (8) and (10), by Lemma 2.2 we know that A is a 3-Lie algebra with respect to the multiplication (4), which leads to (11) and (12), and the remaining brackets are all zero. Let A_ω^δ denote this 3-Lie algebra $(A, [\cdot, \cdot, \cdot]_\omega)$.

Now we show that A_ω^δ is a canonical Nambu 3-Lie algebra. It suffices to show that it can be realized as a canonical Nambu 3-Lie algebra. Choose the basis elements L_r and M_r of A to be

$$L_r = z \exp(rx), \quad M_r = y \exp(-rx), \quad \text{for all } r \in \mathbb{Z},$$

Then the Nambu bracket on A becomes

$$\begin{aligned} [L_r, L_s, M_t]_\partial &= \frac{\partial(L_r, L_s, M_t)}{\partial(x, y, z)} = \begin{vmatrix} rz \exp(rx) & 0 & \exp(rx) \\ sz \exp(sx) & 0 & \exp(sx) \\ -ty \exp(-tx) & \exp(-tx) & 0 \end{vmatrix} = (s - r)L_{r+s-t}, \\ [L_r, M_s, M_t]_\partial &= \frac{\partial(L_r, M_s, M_t)}{\partial(x, y, z)} = \begin{vmatrix} rz \exp(rx) & 0 & \exp(rx) \\ -sy \exp(-sx) & \exp(-sx) & 0 \\ -ty \exp(-tx) & \exp(-tx) & 0 \end{vmatrix} = (t - s)M_{s+t-r}, \\ [M_r, M_s, M_t]_\partial &= \frac{\partial(M_r, M_s, M_t)}{\partial(x, y, z)} = \begin{vmatrix} -ry \exp(rx) & \exp(rx) & 0 \\ -sy \exp(-sx) & \exp(-sx) & 0 \\ -ty \exp(-tx) & \exp(-tx) & 0 \end{vmatrix} = 0, \\ [L_r, L_s, L_t]_\partial &= \frac{\partial(L_r, L_s, L_t)}{\partial(x, y, z)} = \begin{vmatrix} rz \exp(rx) & 0 & \exp(rx) \\ sz \exp(sx) & 0 & \exp(sx) \\ tz \exp(tx) & 0 & \exp(tx) \end{vmatrix} = 0. \end{aligned}$$

Thus the bracket $[\cdot, \cdot]_\omega$ and the Nambu bracket on A are consistent, showing that the 3-Lie algebra A with the bracket $[\cdot, \cdot]_\omega$ is a canonical Nambu 3-Lie algebra.

We next show that A_ω^δ is simple. For every $r, s \in \mathbb{Z}$ and $r \neq s$, from (11) and (12) we obtain that both

$$[L_r, L_s, M_s]_\omega = (s - r)L_r \quad \text{and} \quad [L_s, M_r, M_s]_\omega = (s - r)M_r$$

are in the derived algebra of A_ω^δ . So, A_ω^δ is equal to its derived algebra. Now, let J be a non-zero ideal of A_ω^δ . If $0 \neq u \in J$, then there exist $a_i, b_j \in \mathbb{F}$ such that

$$u = \sum_{i=p}^q a_i L_i + \sum_{j=m}^n b_j M_j, \quad p < q, \quad m < n, \quad a_p \neq 0 \text{ or } b_m \neq 0.$$

Let $r \in \mathbb{Z}$ such that $r > q$ and $r > n$. Then from (11) and (12) we have

$$\text{ad}^l(L_r, M_r)(u) = \sum_{i=p}^q a_i (i - r)^l L_i + \sum_{j=m}^n b_j (j - r)^l M_j \in J, \quad \text{for any positive integer } l.$$

By Vandermonde determinant, we obtain $L_p \in J$ or $M_m \in J$. Therefore, for every $s \in \mathbb{Z}$, $s \neq p$,

$$L_s = \frac{1}{p - s} [L_s, L_p, M_p]_\omega \in J \quad \text{and} \quad M_s = \frac{1}{p - s} [L_p, M_s, M_p]_\omega \in J.$$

From (12), $M_p = \frac{1}{s - p} [L_s, M_p, M_s]_\omega \in J$. Hence $J = A$, showing that A is simple. \square

The 3-Lie algebra structure $[\cdot, \cdot]_\omega$ of A induces different Lie algebra structures on A . For a fixed $k \in \mathbb{Z}$, define the Lie bracket $[\cdot, \cdot]_{L_k}$ on A by

$$[u, v]_{L_k} = [u, v, L_k]_\omega, \quad \text{for all } u, v \in A.$$

Corollary 4.2 *Let $M = \sum_{s \in \mathbb{Z}} \mathbb{F} M_s$ and $L = \sum_{s \in \mathbb{Z}} \mathbb{F} L_s$. Then M is a Lie subalgebra of $(A, [\cdot, \cdot]_{L_k})$ isomorphic to a Witt algebra, and L is a reducible M -module. Furthermore, $\mathbb{F} L_k$ is the center of $(A, [\cdot, \cdot]_{L_k})$.*

Proof. The results follow from $[L_r, L_s]_{L_k} = 0$, as well as

$$[M_r, M_s]_{L_k} = (s - r)M_{r+s-k} \quad \text{and} \quad [L_r, M_s]_{L_k} = (r - k)L_{r+k-s}$$

for all $r, s \in \mathbb{Z}$. \square

Similarly, for each $k \in \mathbb{Z}$, we can define a new Lie bracket on A by

$$[u, v]_{M_k} = [u, v, M_k]_\omega, \quad \text{for all } u, v \in A,$$

a similar result to Corollary 4.2 holds.

5 The inner derivation algebra of A_ω^δ

To describe the structure of $\text{ad}A_\omega^\delta$ and its connection to the representations of Witt algebras, we need a basis of $\text{ad}A_\omega$. For $0 \neq r \in \mathbb{Z}$, let

$$p_r = \frac{1}{2}(\text{ad}(L_0, M_{-r}) + \text{ad}(L_r, M_0)) \quad q_r = \frac{1}{r}(\text{ad}(L_0, M_{-r}) - \text{ad}(L_r, M_0)), \quad (13)$$

$$x_r = \frac{1}{r}\text{ad}(L_r, L_0) \quad z_r = \frac{1}{-r}\text{ad}(M_{-r}, M_0), \quad (14)$$

and let

$$p_0 = \text{ad}(L_0, M_0) \quad q_0 = \text{ad}(L_0, M_0) - \text{ad}(L_1, M_1), \quad (15)$$

$$x_0 = \frac{1}{2}\text{ad}(L_1, L_{-1}) \quad z_0 = \frac{1}{2}\text{ad}(M_1, M_{-1}). \quad (16)$$

Theorem 5.1 *The set*

$$\{p_r, q_r, x_r, z_r \mid r \in \mathbb{Z}\}$$

is a basis of $\text{ad}A_\omega^\delta$, and the multiplication in this basis is as follows: for all $r, s \in \mathbb{Z}$,

$$[p_r, p_s] = (r - s)p_{r+s}, \quad [p_r, q_s] = -sq_{r+s}, \quad (17a)$$

$$[p_r, x_s] = -sx_{r+s}, \quad [p_r, z_s] = -sz_{r+s}, \quad (17b)$$

$$[q_r, q_s] = 0, \quad [q_r, x_s] = -2x_{r+s}, \quad (17c)$$

$$[q_r, z_s] = 2z_{r+s}, \quad [x_r, x_s] = 0, \quad (17d)$$

$$[z_r, z_s] = 0, \quad [z_r, x_s] = q_{r+s}. \quad (17e)$$

Proof. For convenience, write

$$W_{r,s} = \text{ad}(L_r, M_s), \quad X_{r,s} = \text{ad}(L_r, L_s), \quad Y_{r,s} = \text{ad}(M_r, M_s) \quad (18)$$

where $r, s \in \mathbb{Z}$. Then, for $t \in \mathbb{Z}$,

$$W_{r,s}(L_t) = (r - t)L_{t+r-s} \quad \text{and} \quad W_{r,s}(M_t) = (t - s)M_{t+s-r} \quad (19a)$$

$$X_{r,s}(L_t) = 0 \quad \text{and} \quad X_{r,s}(M_t) = (s - r)L_{r+s-t} \quad (19b)$$

$$Y_{r,s}(L_t) = (s - r)M_{r+s-t} \quad \text{and} \quad Y_{r,s}(M_t) = 0. \quad (19c)$$

A tedious calculation yields that the set

$$\{W_{0,0}, W_{1,1}, X_{1,-1}, Y_{1,-1}\} \cup \{W_{r,0}, W_{0,r}, X_{r,0}, Y_{r,0} \mid r \in \mathbb{Z}, r \neq 0\} \quad (20)$$

is linearly independent in $\text{ad}A_\omega^\delta$.

Next, we show that (20) is a basis of $\text{ad}A_\omega^\delta$. By (19a), if $s = r$, we have $W_{r,s} = (1 - r)W_{0,0} + rW_{1,1}$; if $s \neq r$, then

$$W_{r,s} = \frac{1}{r - s}(rW_{r-s,0} - sW_{0,s-r}).$$

From (19b) and (19c), if $s = -r$, then $X_{r,s} = rX_{1,-1}$ and $Y_{r,s} = rY_{1,-1}$; if $s \neq -r$, then we obtain

$$X_{r,s} = \frac{r-s}{r+s}X_{r+s,0} \quad \text{and} \quad Y_{r,s} = \frac{r-s}{r+s}Y_{r+s,0}.$$

Also, $X_{r,s} = -X_{s,r}$ and $Y_{r,s} = -Y_{s,r}$, for all $r, s \in \mathbb{Z}$. Thus, the elements in (20) form a basis of $\text{ad}A_\omega^\delta$.

From the definitions of p_r, q_r, x_r, z_r given in (13)–(16), we know that

$$\begin{aligned} p_0 &= W_{0,0} & q_0 &= W_{0,0} - W_{1,1} \\ x_0 &= \frac{1}{2}X_{1,-1} & z_0 &= \frac{1}{2}Y_{1,-1} \end{aligned}$$

and, for $0 \neq r \in \mathbb{Z}$,

$$\begin{aligned} p_r &= \frac{1}{2}(W_{r,0} + W_{0,-r}) & q_r &= \frac{1}{r}(W_{0,-r} - W_{r,0}) \\ x_r &= \frac{1}{r}X_{r,0} & z_r &= \frac{1}{-r}Y_{-r,0}. \end{aligned}$$

Therefore, $\{p_r, q_r, x_r, z_r\}$ is a basis of $\text{ad}A_\omega^\delta$.

To obtain the multiplication in (17a)–(17e), we have to calculate the Lie brackets of any two elements in (18). Note that if a bracket is zero, we omit it.

From (19a) it follows that, for all $r, s \in \mathbb{Z}$,

$$\begin{aligned} [W_{r,0}, W_{s,0}] &= (r-s)W_{r+s,0} & [W_{0,r}, W_{0,s}] &= (s-r)W_{0,r+s} \\ [W_{1,1}, W_{r,0}] &= -rW_{r,0} & [W_{1,1}, W_{0,s}] &= sW_{0,s} \\ [W_{0,0}, W_{r,0}] &= -rW_{r,0} & [W_{0,0}, W_{0,s}] &= sW_{0,s} \end{aligned}$$

and

$$[W_{r,0}, W_{0,s}] = \begin{cases} \frac{1}{r-s}(r^2W_{r-s,0} - s^2W_{0,s-r}), & r \neq s, \\ 2rW_{0,0} + r^2(W_{1,1} - W_{0,0}), & r = s. \end{cases}$$

Using (19a) and (19b), we obtain

$$\begin{aligned} [X_{r,0}, W_{0,s}] &= rX_{r-s,0} & [X_{1,-1}, W_{0,s}] &= 2X_{-s,0} \\ [X_{r,0}, W_{0,0}] &= rX_{r,0} & [X_{1,-1}, W_{r,0}] &= -2X_{r,0} \\ [X_{r,0}, W_{1,1}] &= (r-2)X_{r,0} & [X_{1,-1}, W_{1,1}] &= -2X_{1,-1} \end{aligned}$$

and

$$[X_{r,0}, W_{s,0}] = \begin{cases} r^2X_{1,-1}, & s = -r, \\ \frac{r(r-s)}{r+s}X_{r+s,0}, & s \neq -r. \end{cases}$$

It follows from (19a) and (19c) that

$$\begin{aligned} [Y_{r,0}, W_{s,0}] &= -rY_{r-s,0} & [Y_{1,-1}, W_{0,s}] &= 2Y_{s,0} \\ [Y_{r,0}, W_{0,0}] &= -rY_{r,0} & [Y_{1,-1}, W_{r,0}] &= -2Y_{-r,0} \\ [Y_{r,0}, W_{1,1}] &= (2-r)Y_{r,0} & [Y_{1,-1}, W_{1,1}] &= 2Y_{1,-1} \end{aligned}$$

and

$$[Y_{r,0}, W_{0,s}] = \begin{cases} -r^2 Y_{1,-1}, & s = -r, \\ \frac{r(s-r)}{r+s} Y_{r+s,0}, & s \neq -r. \end{cases}$$

The following are obtained from (19b) and (19c)

$$\begin{aligned} [X_{r,0}, Y_{1,-1}] &= 2(W_{r,0} - W_{0,-r}) \\ [X_{1,-1}, Y_{s,0}] &= -2(W_{-s,0} - W_{0,s}) \\ [X_{1,-1}, Y_{1,-1}] &= -4(W_{0,0} - W_{1,1}) \\ [X_{r,0}, Y_{s,0}] &= \begin{cases} \frac{rs}{r-s}(W_{r-s,0} - W_{0,s-r}), & r \neq s, \\ r^2(W_{1,1} - W_{0,0}), & r = s, \end{cases} \end{aligned}$$

So, the multiplication of the basis elements of $\{p_r, q_r, x_r, z_r \mid r \in \mathbb{Z}\}$ satisfies (17a)–(17e). \square

We are now in a position to study the structure of $\text{ad}A_\omega^\delta$ and its connection to the representations of Witt algebras. Define four subspaces of $\text{ad}A_\omega^\delta$ by

$$W = \sum_{r \in \mathbb{Z}} \mathbb{F}p_r, \quad V_1 = \sum_{r \in \mathbb{Z}} \mathbb{F}q_r, \quad V_2 = \sum_{r \in \mathbb{Z}} \mathbb{F}x_r, \quad V_3 = \sum_{r \in \mathbb{Z}} \mathbb{F}z_r.$$

Theorem 5.2 *If $\text{ch } \mathbb{F} \neq 2$, then*

(a) *W is a simple Lie algebra isomorphic to a Witt algebra, and $V = V_1 \oplus V_2 \oplus V_3$ is a Lie algebra isomorphic to the simple Lie algebra $sl(2, \mathbb{F}) \otimes \mathbb{F}[t, t^{-1}]$.*

(b) *The inner derivation algebra $\text{ad}A_\omega^\delta$ is an indecomposable Lie algebra, and it can be decomposed into the semi-direct sum of subalgebras W and V . Moreover, V is the only minimal ideal of $\text{ad}A_\omega^\delta$.*

(c) *For $i = 1, 2, 3$, define ρ_i to be the map from W to V_i by*

$$\rho_i(p_r) = \begin{cases} q_r, & \text{if } i = 1, \\ x_r, & \text{if } i = 2, \\ z_r, & \text{if } i = 3, \end{cases}$$

where $r \in \mathbb{Z}$. Then (V_i, ρ_i) is an irreducible intermediate series W -module isomorphic to the regular representation of W .

Proof. The Lie algebra $W = \sum_{r \in \mathbb{Z}} \mathbb{F}p_r$ is isomorphic to a Witt algebra, since $[p_r, p_s] = (r-s)p_{r+s}$ for all $r, s \in \mathbb{Z}$.

We next show that the Lie algebra V is isomorphic to $sl(2, \mathbb{F}) \otimes \mathbb{F}[t, t^{-1}]$. Let $sl(2, \mathbb{F}) = \mathbb{F}e \oplus \mathbb{F}h \oplus \mathbb{F}f$. Define the map $\sigma : V \rightarrow sl(2, \mathbb{F}) \otimes \mathbb{F}[t, t^{-1}]$ by

$$\sigma(q_r) = h \otimes t^r, \quad \sigma(z_r) = e \otimes t^r, \quad \sigma(x_r) = f \otimes t^r, \quad \text{for all } r \in \mathbb{Z}.$$

From Theorem 5.1 it follows that

$$\begin{aligned}
[\sigma(q_r), \sigma(q_s)] &= \sigma([q_r, q_s]) = 0, \\
[\sigma(x_r), \sigma(x_s)] &= \sigma([x_s, x_s]) = 0, \\
[\sigma(z_r), \sigma(z_s)] &= \sigma([z_r, z_s]) = 0, \\
[\sigma(q_r), \sigma(z_s)] &= [h \otimes t^r, e \otimes t^s] = 2e \otimes t^{r+s} = 2\sigma(z_{r+s}) = \sigma([q_r, z_s]), \\
[\sigma(q_r), \sigma(x_s)] &= [h \otimes t^r, f \otimes t^s] = -2f \otimes t^{r+s} = 2\sigma(x_{r+s}) = \sigma([q_r, x_s]), \\
[\sigma(z_r), \sigma(x_s)] &= [e \otimes t^r, f \otimes t^s] = h \otimes t^{r+s} = \sigma(q_{r+s}) = \sigma([z_r, x_s]).
\end{aligned}$$

This proves result (a). The result (b) is clear.

We now prove (c). It suffices to show the result for $i = 1$, since the other two cases are similar. The identity (17a) shows that V_1 is an irreducible W -module. This module is an intermediate series module because

$$V_1 = \bigoplus_{r \in \mathbb{Z}} V_{\lambda_r},$$

where each weight space $V_{\lambda_r} = \mathbb{F}q_r$ of weight $-r$ is of dimension 1 by identity (17a). From the definitions of W and V_1 , it is easily seen that ρ_1 is an isomorphism, so (V_1, ρ) is isomorphic to the regular representation of the Witt algebra W . \square

6 Representations of A_ω^δ and $\text{ad}A_\omega^\delta$

We first characterize the regular representation of the 3- Lie algebra A_ω^δ .

Theorem 6.1 *Let $V = A$ as a vector space. The regular representation (V, ad) of the 3-Lie algebra A_ω^δ is a Harish-Chandra module.*

Proof. Since $V = A$ as a vector space, $V = \bigoplus_{r \in \mathbb{Z}} (\mathbb{F}L_r \oplus \mathbb{F}M_r)$. By Theorem 4.1 we have that the Cartan subalgebra of A_ω^δ is

$$H = \mathbb{F}L_0 \oplus \mathbb{F}M_0,$$

and

$$\text{ad}(L_0, M_0)(L_t) = -tL_t \quad \text{and} \quad \text{ad}(L_0, M_0)(M_t) = tM_t \quad \text{for all } t \in \mathbb{Z},$$

If $t \in \mathbb{Z}$, define a linear function $\lambda_t : H \wedge H \rightarrow \mathbb{F}$ by

$$\lambda_t(L_0, M_0) = -t.$$

A simple calculation yields that $V = \bigoplus_{\lambda_t \in (H \wedge H)^*} V_{\lambda_t}$ where

$$V_{\lambda_t} = \mathbb{F}L_t \oplus \mathbb{F}M_{-t}$$

are the weight spaces of dimension 2. The desired result follows. \square

We then describe the natural representation of the inner derivation algebra $\text{ad}A_\omega^\delta$. Let V be the vector space A over \mathbb{F} . Recall that the natural module (V, ρ) of $\text{ad}A_\omega^\delta$ on V is defined by

$$\rho(D)(z) = D(z) \quad \text{for all } D \in \text{ad}A_\omega^\delta, z \in V.$$

Theorem 6.2 *The natural module (V, ρ) of the derivation algebra $\text{ad}A_\omega^\delta$ is an intermediate series module.*

Proof. From Theorem 5.1 it follows that $H = \mathbb{F}p_0 \oplus \mathbb{F}q_0$ is the Cartan subalgebra of the Lie algebra $\text{ad}A_\omega^\delta$. Under the natural action of inner derivations, for all $t \in \mathbb{Z}$, by (19a) and (15) we obtain

$$p_0(L_t) = -tL_t, \quad p_0(M_t) = tM_t, \quad q_0(L_t) = -L_t, \quad \text{and} \quad q_0(M_t) = M_t.$$

For each $t \in \mathbb{Z}$, define a linear map $\lambda_t : H \wedge H \rightarrow \mathbb{F}$ by $\lambda_t(p_0) = t$ and $\lambda_t(q_0) = 1$. Then we obtain weight spaces

$$V_{\lambda_t} = \mathbb{F}M_t \quad \text{and} \quad V_{-\lambda_t} = \mathbb{F}L_t.$$

Furthermore, $V = \bigoplus_{t \in \mathbb{Z}} (V_{\lambda_t} \oplus V_{-\lambda_t})$. But then $\dim V_{\lambda_t} = \dim V_{-\lambda_t} = 1$ for all $t \in \mathbb{Z}$, so the natural module (V, ρ) is an intermediate series module of $\text{ad}A_\omega^\delta$. \square

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